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The optimal form of the fractional-order difference feedbacks in enhancing the stability of a sdof vibration system

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A B S T R A C T

Many practical systems including the slightly damped mechanical systems, even they are already stable, are required to be controlled, in order to get better performance or better stability. In this paper, the concept of fractional-order difference feedback that generalizes the displacement difference feedback, velocity difference feedback and acceleration difference feedback, is proposed for improving the stability of a sdof vibration system. It is found that among the various state difference feedbacks, some fractional-order difference feedbacks including fractional-order integrators and fractional-order differentiators improve the stability of vibration systems best. Fractional-order integrator/differentiator is a controller with memory for the whole time history, its implementation is usually more complicated than the classical PID control and acceleration control. Thus, proper classical controller is suggested for improving the stability of the vibration system with small damping and small delay. If a displacement sensor is used, then the optimal form of state difference feedbacks for enhancing stability is the displacement difference feedback with $k>0$. If an acceleration sensor is used, then the optimal form of state difference feedbacks for enhancing stability is the acceleration difference feedback with $k<0$. Moreover, on the basis of the principal of stability switch, the admissible feedback gains and delay governing the asymptotical stability and γ-stability are studied in detail, and illustrated with numerical experiments.

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1. Introduction

Vibration control is a key problem in many practical systems. Various control schemes can be used to stabilize an unstable motion [1]. Even for some stable systems such as slightly damped mechanical systems including rotary crane [2] and flexible link [3], their free vibration decays very slowly, thus a control, say a negative velocity feedback, can be used to enhance the system stability so that the closed-loop arrives at its steady states more quickly. The velocity sensors, however, may result in cost, space, and malfunction problems. Measuring velocity from displacement sensors usually introduces heavy noise and deviation, which have to be removed by using additional filters [4]. To overcome such disadvantages, the state difference feedback that uses only the difference between the present state $x(t)$ and the past state $x(t-\tau)$, $(\tau>0)$, which was originally proposed by Pyragas [5] for controlling chaos and can be implemented easily in applications, will be useful [2,6]. This feedback method is peculiarly useful, considering exact information about the steady state is usually unavailable and the control is robust against parameter changes. In [7], it is shown that under certain mild conditions,

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a state difference feedback can always stabilize a system with even number of unstable modes. In addition, on the basis of stability switches [8], a new criterion is established in [9] for the stabilization problem of unstable vibration systems via state difference feedback, and an effective and constructive procedure for determining the admissible values of the feedback gains and the delay is given.

To show the ability in improving system stability of displacement difference feedback, let us consider the vibration system of single degree of freedom (sdof) *in dimensionless form*

$$\ddot{x}(t) + 2\xi\dot{x}(t) + x(t) = 0 \tag{1}$$

the models discussed in [2,10–12] fall into this category. The zero solution $x(t) \equiv 0$ is asymptotically stable if $\xi > 0$. Here the stability of a system is improved, we mean that the real part of the rightmost root(s) of the closed-loop is less than that of the control plant, namely the rightmost root(s) is shifted in the complex plane from right to left. We consider the cases with under-damped term, namely $0 < \xi < 1$, then the characteristic roots read

$$\lambda_{1,2} = -\xi \pm \sqrt{1 - \xi^2}\,\mathrm{i}, \quad (\mathrm{i}^2 = -1)$$

When a displacement difference feedback $u = -k(x(t) - x(t-\tau))$ with $k > 0$ is performed to the vibration system, the closed-loop system reads

$$\ddot{x}(t) + 2\xi\dot{x}(t) + x(t) = -k(x(t) - x(t-\tau)) \tag{2}$$

For small $\tau > 0$, one has $\ddot{x}(t) + 2\xi\dot{x}(t) + x(t) \approx -k\tau\dot{x}(t)$, thus the two dominate roots among the infinite number of characteristic roots become

$$\lambda^c_{1,2} \approx -\left(\xi + \frac{k\tau}{2}\right) \pm \sqrt{1 - \left(\xi + \frac{k\tau}{2}\right)^2}\,\mathrm{i}$$

Because $\mathrm{Re}(\lambda^c_{1,2}) < \mathrm{Re}(\lambda_{1,2})$, where $\mathrm{Re}(z)$ stands for the real part of z, thus the stability of the vibration system has been enhanced if $k > 0$, and it has been deteriorated if $k < 0$.

On the other hand, acceleration sensors are actually most widely used for vibration control, because acceleration sensors measure acceleration easily and accurately [11–16]. Thus, a delayed acceleration feedback, as well as a delayed velocity feedback measuring from acceleration sensors, can be also used to stabilize unstable motion or to enhance system stability. In addition, in machine tool dynamics, the displacement difference feedback and the velocity difference feedback are also called regenerative cutting force and regenerative damping force respectively [17,18]. Chatter instability in machining process, due to regeneration of surface waviness, has been shown to involve regenerative cutting/damping forces [19]. This form of instability causes an unacceptable surface finish, along with excessive tool wear or breakage, thereby limiting the metal removal rate that can be achieved [20]. Enhancement of dynamic stability is very important in machining dynamics.

Since the pioneering work [21,22] of Bagley and Torvik, who used the $\frac{1}{2}$-order derivative or $\frac{3}{2}$-order derivative to describe damping in an immersed plate in a Newtonian fluid and a gas in a fluid respectively, it has found many applications of fractional calculus [23–26]. Analysis shows that fractional calculus provides better results than classical calculus [23]. In control theory, the idea of using fractional-order controllers belongs to Oustaloup, who developed the so-called CRONE controllers. The $PI^\lambda D^\mu$-controller involving an integrator of order λ and differentiator of order μ was proposed by Podlubny. A fractional-order controller is more flexible and it provides better control effect and better performance than an integer-order controller [23]. Because displacement, velocity and acceleration can be considered as the zero-order, first-order and second-order derivatives of the displacement, it is natural to investigate the control effect of a real-order difference feedback in the form

$$u = -k(D^\alpha x(t) - D^\alpha x(t-\tau)) \quad (\alpha \in \mathbb{R}) \tag{3}$$

where D^α stands for fractional-order derivative/integration [23–26]. We call this controller *fractional-order difference feedback*.

With such a fractional-order controller, the first question comes:

- For what values of α, the fractional-order controllers with a small delay, including displacement difference feedback, velocity difference feedback, and acceleration difference feedback, improve the stability of the vibration system best?

For $\tau > 0$ that is not small, the approximation $\ddot{x}(t) + 2\xi\dot{x}(t) + x(t) \approx -k\tau\dot{x}(t)$ is not guaranteed, so it is not clear whether such a feedback control improves the system stability or not. This is the case for the fractional-order controller given in Eq. (3). Thus, there comes the second question:

- How to find a region D in the first quadratic of (k, α, τ)-coordinate such that the system stability is improved, namely the corresponding conjugate rightmost characteristic roots have smaller negative real parts for all $(k, \alpha, \tau) \in D$?

Or more generally, we need to know

- How to find a region D in the first quadratic of (k,α,τ)-coordinate such that the system stability is improved to γ-stability, namely the corresponding conjugate rightmost characteristic roots have real part less than a given negative number $-\gamma$ for all $(k,\alpha,\tau)\in D$?

The primary objective of this paper is to present an answer to the three problems. We begin in Section 2 with a brief introduction of the fractional difference feedback. Next in Section 3, a comparison study of the state difference feedback in different forms is made. Then in Section 4, on the base of stability switches, it is shown the parameter region for improving the stability of the system is governed by two branches associated with k and τ, and this parameter region is given analytically and explicitly. In Section 5, the problem of improving stability to γ-stability is investigated, a method is proposed for determining the admissible delay, and it is illustrated with a numerical experiment. Finally, some conclusions are drawn in Section 6.

2. A fractional-order state difference feedback

The fractional-order difference feedback defined by

$$u=-k(D^{\alpha}x(t)-D^{\alpha}x(t-\tau))\quad(\alpha\in\mathbb{R}) \tag{4}$$

involves fractional-order derivative ($\alpha>0$) or fractional-order integration ($\alpha<0$), which have different definitions. The most popular definitions are Riemann–Liouville definition, Grünwald–Letnikov definition and Caputo definition [23]. Usually, the Riemann–Liouville definition is widely used for problem description because it requires less constraints on the state variables; while the Grünwald–Letnikov definition is preferable for numerical computation because it is a summation of state differences; and the Caputo definition is preferable for control problems because it admits more operational rules that are analog to the ones for classical calculus than the other definitions. To begin with the comparison study, it is necessary to make clear the meaning of real-order integration and real-order differentiation. Here in this paper, we use Caputo's definition of real-order derivative/integration.

Let $\alpha\in\mathbb{R}$, and $m=[\alpha]+1$, where $[\alpha]$ stands for the greatest integer that is not larger than α, then

$$D^{\alpha}x(t)=\frac{1}{\Gamma(\alpha-m)}\int_0^t\frac{x^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\,\mathrm{d}\tau \tag{5}$$

where $\Gamma(z)$ is Euler gamma function, defined by

$$\Gamma(z)=\int_0^{\infty}\mathrm{e}^{-t}t^{z-1}\,\mathrm{d}t\quad(\mathrm{Re}(z)>0)$$

and $x^{(m)}(t)$ is the classical derivative/integration, defined by

$$x^{(m)}(t)=\begin{cases}\dfrac{\mathrm{d}^m x(t)}{\mathrm{d}t^m}, & m\in\mathbb{N}\\ x(t), & m=0\\ x^{(-n)}(t)=\dfrac{1}{\Gamma(n)}\int_0^t\dfrac{x(\tau)}{(t-\tau)^{1-n}}\mathrm{d}\tau, & m=-n,\ n\in\mathbb{N}\end{cases}$$

One of the important features of Caputo's fractional-order derivative is the following formula for its Laplace transformation

$$\mathscr{L}(D^{\alpha}x(t))=s^{\alpha}\mathscr{L}(x(t))-\sum_{i=1}^{m-1}s^{\alpha-i-1}x^{(i)}(0) \tag{6}$$

With zero initial conditions, one has simply $\mathscr{L}(D^{\alpha}x(t))=s^{\alpha}\mathscr{L}(x(t))$.

3. The optimal order of the fractional state difference feedbacks for improving stability

To make the exposition as simple as possible, the sdof vibration system is addressed in this paper. Under the fractional-order difference feedback (4), the closed-loop reads

$$\ddot{x}(t)+2\xi\dot{x}(t)+x(t)=-k(D^{\alpha}x(t)-D^{\alpha}x(t-\tau)) \tag{7}$$

Using Laplace transformation one finds the characteristic function of the closed-loop as follows:

$$p(s)=s^2+2\xi s+1+ks^{\alpha}(1-\mathrm{e}^{-s\tau}) \tag{8}$$

Let $s=\lambda-\xi$, one has

$$f(\lambda)=p(\lambda-\xi)=\lambda^2-\xi^2+1+k(\lambda-\xi)^{\alpha}(1-\mathrm{e}^{-(\lambda-\xi)\tau}) \tag{9}$$

Then the stability is improved if the rightmost roots of $f(\lambda)$ have real part less than 0. Obviously, when $\tau = 0$, the two roots of $f(\lambda) = 0$ stay on the imaginary axis with $\lambda = \pm\sqrt{1-\xi^2}\mathrm{i}$. It is required to figure out the sign of

$$\Lambda = \mathrm{Re}\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right]_{\tau=0,\,\lambda=\pm\sqrt{1-\xi^2}\mathrm{i}}$$

If $\Lambda > 0$, then $f(\lambda)$ admits one new pair conjugate roots with positive real part, and if $\Lambda < 0$, then $f(\lambda)$ admits one new pair of conjugate roots with negative real part, as τ increases from 0. In the latter case, with a small delay in the control path, the stability of the sdof vibration system is improved. The smaller the Λ is, the better the stability of the closed-loop is. When $\xi > 1/\sqrt{2}$, a state feedback may not be necessary for enhancing the system stability, because the damping coefficient may be large enough for system stability. Thus, the case of $\xi \in (0, 1/\sqrt{2})$ will be addressed below.

It is easy to know that $\Lambda = -k/2$ if $\alpha = 0$, and $\Lambda = k(1-4\xi^2)/2$ if $\alpha = 2$. Thus for $k > 0$, the displacement difference feedback improves the stability of the vibration system, but the acceleration difference feedback does not. Hence, the displacement difference feedback is better than the acceleration difference feedback, if $k > 0$. When $k < 0$, the acceleration difference feedback improves the stability of the vibration system, but the displacement difference feedback does not. Hence, the acceleration difference feedback is better than the displacement difference feedback, if $k < 0$.

In general, straightforward differentiation on $f(\lambda) = 0$ with respect to τ gives

$$\Lambda = -\frac{k\Delta}{2\sqrt{1-\xi^2}}$$

where

$$\begin{aligned}\Delta &= \sqrt{1-\xi^2}\cos\left(\alpha\arctan\frac{\sqrt{1-\xi^2}}{\xi} - \alpha\pi\right) + \xi\sin\left(\alpha\arctan\frac{\sqrt{1-\xi^2}}{\xi} - \alpha\pi\right)\\ &= \sin\left((1+\alpha)\arctan\frac{\sqrt{1-\xi^2}}{\xi} - \alpha\pi\right)\end{aligned}$$

It follows that

$$\Lambda = -\frac{k\sin\left((1+\alpha)\arctan\dfrac{\sqrt{1-\xi^2}}{\xi} - \alpha\pi\right)}{2\sqrt{1-\xi^2}} \tag{10}$$

Moreover, let

$$\alpha_j^+ \stackrel{\text{def}}{=} \frac{\arctan\dfrac{\sqrt{1-\xi^2}}{\xi} - \dfrac{\pi}{2} - 2j\pi}{\pi - \arctan\dfrac{\sqrt{1-\xi^2}}{\xi}} \quad (j = 0, \pm 1, \pm 2, \ldots) \tag{11}$$

$$\alpha_l^- \stackrel{\text{def}}{=} \frac{\arctan\dfrac{\sqrt{1-\xi^2}}{\xi} + \dfrac{\pi}{2} - 2l\pi}{\pi - \arctan\dfrac{\sqrt{1-\xi^2}}{\xi}} \quad (l = 0, \pm 1, \pm 2, \ldots) \tag{12}$$

then

$$\min_{\alpha\in\mathbb{R}} \Lambda = \begin{cases} -\dfrac{k}{2\sqrt{1-\xi^2}} & \text{if } \alpha = \alpha_j^+,\ k > 0 \\ -\dfrac{|k|}{2\sqrt{1-\xi^2}} & \text{if } \alpha = \alpha_l^-,\ k < 0 \end{cases} \tag{13}$$

That is to say, for a fixed $\xi \in (0,1)$ and a small τ, the fractional-order difference feedback (3) improves the stability of the vibration system best if $k > 0$ and $\alpha = \alpha_j^+$, or $k < 0$ and $\alpha = \alpha_l^-$. A general view on the α that minimizes Λ is given in Fig. 1, and some values of α_j that minimize Λ for fixed ξ is given in Table 1.

In summary, the following theorem answers our first question.

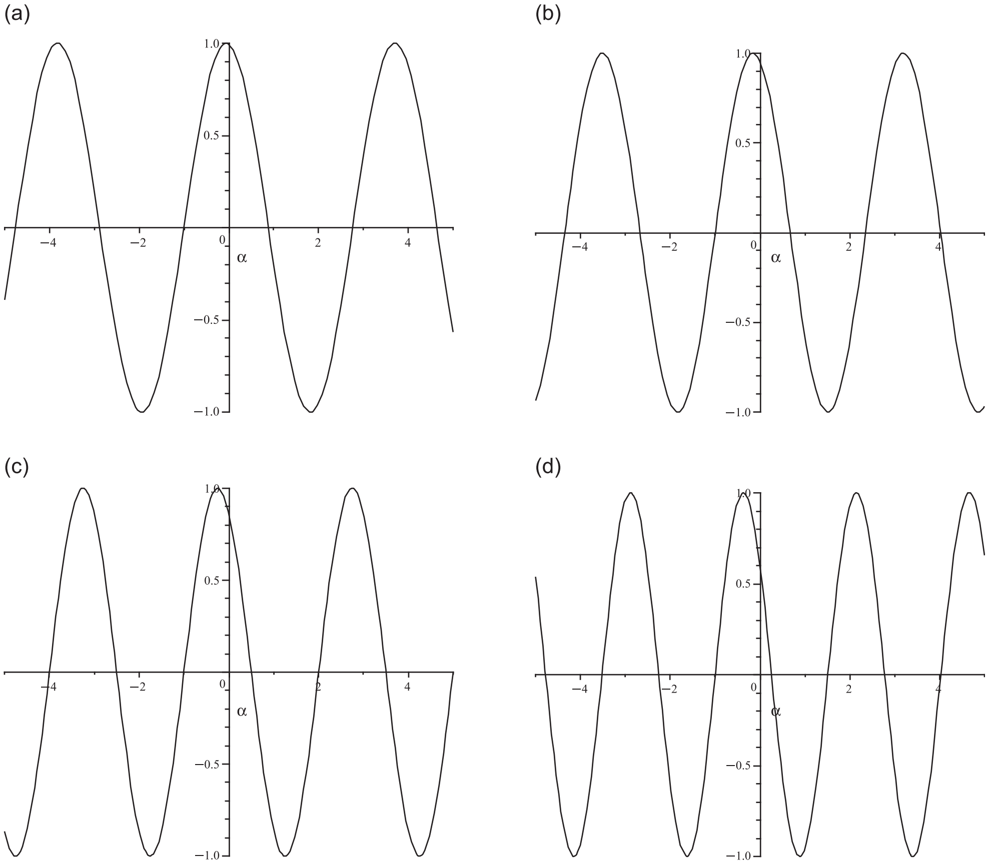

Fig. 1. The plot of Λ vs. α for (a) $\xi = 0.1$; (b) $\xi = 0.3$; (c) $\xi = 0.5$; (d) $\xi = 0.8$.

Table 1
Some values of α_j^+, α_l^- that minimize Λ defined in Eq. (10).

ξ	$k>0$			$k<0$		
	α_1^+	α_0^+	α_{-1}^+	α_1^-	α_0^-	α_{-1}^-
0.05	−3.907	−0.0309	3.846	−1.969	1.907	5.784
0.1	−3.820	−0.0599	3.700	−1.940	1.820	5.580
0.3	−3.513	−0.1624	3.188	−1.838	1.513	4.863
0.5	−3.250	−0.2500	2.750	−1.750	1.250	4.250
0.8	−2.886	−0.3712	2.144	−1.629	0.8864	3.402

Theorem 1. *Among the real-order feedback controllers $-k(D^{\alpha}x(t) - D^{\alpha}x(t-\tau))$ with small $\tau > 0$, including the classical integral feedback, displacement feedback, velocity feedback and acceleration feedback, the fractional-order α integrator/differentiator with $\alpha = \alpha_j^+$ or $\alpha = \alpha_l^-$ improves the stability of the sdof vibration system best for $k>0$ or $k<0$ respectively.*

Remark 1. From Eqs. (11) and (12), it can be seen that for small $\xi > 0$, $\alpha_0^+ \approx 0$ and $\alpha_0^- \approx 2$. Considering the fact that the realization of a fractional-order controller is usually more difficult than a classical controller, the displacement difference feedback is preferable if $k>0$, and the acceleration difference feedback is preferable if $k<0$, in enhancing the stability of the vibration system.

4. The region *D* in $(\boldsymbol{k}, \tau)$-plane for improving stability

From the above discussion, for $\xi \in (0, 1/\sqrt{2})$, the stability of the vibration system should be improved by using the displacement difference feedback with $k>0$ or by using the acceleration difference feedback with $k<0$. In this section, the second question proposed in Introduction, namely finding a region D in (k, τ)-plane for improving stability of the vibration system with time-invariant $\tau > 0$, will be solved for $k > 0$, by means of stability switches [9]. When the delay is time-variant, the stability analysis of time-delay system is usually carried out by using LMI method, see for example [27,28].

Under the displacement difference feedback, the closed-loop is

$$\ddot{x}(t) + 2\xi\dot{x}(t) + x(t) = -k(x(t) - x(t-\tau)) \tag{14}$$

and its characteristic function reads $p(s) = s^2 + 2\xi s + 1 + k(1 - \mathrm{e}^{-s\tau})$, and consequently

$$f(\lambda) = p(\lambda - \xi) = \lambda^2 - \xi^2 + 1 + k(1 - \mathrm{e}^{-(\lambda-\xi)\tau})$$

Straightforward computation gives

$$\mathrm{Re}\left[\frac{\mathrm{d}\lambda}{\mathrm{d}k}\right]_{k=0} = -\frac{\sin(\sqrt{1-\xi^2}\tau)}{2\sqrt{1-\xi^2}} < 0$$

Here $\sqrt{1-\xi^2}\tau < \pi$ is assumed true (the reason will be given in Remark 2 at the end of this section). Together with $\Lambda = -k/2 < 0$, it means that as τ and k increases from 0, the conjugate rightmost roots of $f(\lambda)$ have real part less than 0, or equivalently, the conjugate rightmost roots of $p(s)$ have real part less than $-\xi$. Thus, the system stability is improved if $\tau > 0$ and $k > 0$ are small enough. Because the root λ of $f(\lambda) = 0$ depends continuously on $\tau > 0$ and $k > 0$, the two parameters will arrive at the boundary between stability and instability if $f(\pm\mathrm{i}\omega) = 0$ has a solution $\omega \geq 0$.

Now, separating the real and imaginary parts of $f(\mathrm{i}\omega) = 0$ gives

$$\begin{cases} -\omega^2 - \xi^2 + 1 + k(1 - \mathrm{e}^{\tau\xi}\cos(\tau\omega)) = 0 \\ k\mathrm{e}^{\tau\xi}\sin(\tau\omega) = 0 \end{cases}$$

Thus, (k, τ) and ω must satisfy

$$\omega\tau = n\pi, \quad -\omega^2 - \xi^2 + 1 + k(1 - (-1)^n\mathrm{e}^{\tau\xi}) = 0$$

for $n = 0, 1, 2, \ldots$. Let

$$g(k,\tau,n) = -\left(\frac{n\pi}{\tau}\right)^2 - \xi^2 + 1 + k - (-1)^n k\mathrm{e}^{\tau\xi} \tag{15}$$

Obviously, $g(k,\tau,0)$ reads

$$g(k,\tau,0) = -\xi^2 + 1 + k - k\mathrm{e}^{\tau\xi}$$

It has exactly one root $k_0 = (1-\xi^2)/(1-\mathrm{e}^{\tau\xi}) \in \mathbb{R}^+$ for any fixed $\tau > 0$, and it has also exactly one root $\tau_0 = (\ln((k+1-\xi^2)/k))/\xi > 0$ because for fixed $k > 0$, $g(k,\tau,0)$ decreases monotonously with τ and satisfies

$$g(k,0,0) = -\xi^2 + 1 > 0, \quad \lim_{\tau\to+\infty} g(k,\tau,0) = -\infty$$

When $n = 2m$ $(m = 1, 2, \ldots)$, one has

$$g(k,\tau,2m) = -\left(\frac{2m\pi}{\tau}\right)^2 - \xi^2 + 1 + k - k\mathrm{e}^{\tau\xi}$$

With a fixed $k \in \mathbb{R}^+$, let

$$T_{2m} = \frac{3}{\xi}W_0\left(\frac{2}{3}\xi\sqrt[3]{\frac{m^2\pi^2}{k\xi}}\right) > 0$$

where $W_0(x)$, which can be calculated directly by using the calculator in the popular softwares such as MAPLE, is the principal branch of Lambert W function, defined as the solution $w = w(x)$ of $w\mathrm{e}^w = x$. Then differentiating $g(k,\tau,2m)$ with respect to τ gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}g(k,\tau,2m) = 8\frac{m^2\pi^2}{\tau^3} - k\xi\mathrm{e}^{\xi\tau}\begin{cases} >0, & \tau \in (0, T_{2m}) \\ <0, & \tau \in (T_{2m}, +\infty) \end{cases}$$

It follows that $g(k,\tau,2m)$ has exactly two positive roots if $g(k,T_{2m},2m) > 0$, it has a repeated root if $g(k,T_{2m},2m) = 0$, and it has no real root if $g(k,T_{2m},2m) < 0$, because

$$\lim_{\tau\to 0} g(k,\tau,2m) = -\infty, \quad \lim_{\tau\to\infty} g(k,\tau,2m) = -\infty$$

In the first two cases, $g(k,\tau,2m) = 0$ has no more than two solutions, the smaller one is denoted by τ_{2m}. Because $g(k,\tau,2\,(m+1)) < g(k,\tau,2m)$, so τ_2 is the smallest one among τ_{2m} $(m = 1, 2, \ldots)$ for given k.

If $n = 2m + 1$ ($m = 0, 1, 2, \ldots$), one has

$$g(k, \tau, 2m+1) = -\left(\frac{(2m+1)\pi}{\tau}\right)^2 - \xi^2 + 1 + k + ke^{\tau\xi}$$

And $g(k, \tau, 2m+1) = 0$ has a unique solution denoted by τ_{2m+1} for any given $k \in \mathbb{R}^+$, since $g(k, \tau, 2m+1)$ increases monotonously with τ, and

$$\lim_{\tau \to +0} g(k, \tau, 2m+1) = -\infty, \quad \lim_{\tau \to +\infty} g(k, \tau, 2m+1) = +\infty$$

Moreover, one has $g(k, \tau, 2(m+1)+1) < g(k, \tau, 2m+1)$, so τ_1 is the smallest one among τ_{2m+1} ($m = 0, 1, \ldots$) for given k.

From the definitions of $g(k, \tau, 2)$ and $g(k, \tau, 1)$, one has $g(k, \tau, 2) < g(k, \tau, 1)$. So $g(k, \tau_2, 1) > g(k, \tau_2, 2) = 0$, if τ_2 does exist. Because $g(k, \tau, 1)$ is increasing in $\tau > 0$, one has $\tau_2 > \tau_1$. In addition, τ_1 depends decreasingly on $k > 0$, because

$$\frac{d\tau(k)}{dk} = -\frac{\tau^3(1 + e^{\xi\tau})}{2\pi^2 + k\xi\tau^3 e^{\xi\tau}} < 0$$

Therefore for any given $k \in \mathbb{R}^+$, one finds the delay bound τ^* from the roots of $g(k, \tau, 0)$, $g(k, \tau, 1)$, namely

$$\tau^* = \min\{\tau_0, \tau_1\} = \min\left\{\frac{\ln((k + 1 - \xi^2)/k)}{\xi}, \tau_1\right\} \tag{16}$$

Alternatively, one can also find the bound k^* of feedback gain for fixed τ. In summary, one has

Theorem 2. *Under the negative state difference feedback* $-k(x(t) - x(t-\tau))$, ($k > 0$), *the region D in* (k, τ)*-plane that improves the stability of the vibration system is given by*

$$D = \{(k, \tau) | k > 0, \tau > 0, g(k, \tau, 0) > 0, g(k, \tau, 1) < 0\}$$

Remark 2. From $g(k, \tau, 1) = 0$, it is easy to know that

$$\lim_{k \to +0} \tau(k) = \pi/\sqrt{1 - \xi^2}, \quad \lim_{k \to +\infty} \tau(k) = 0$$

Hence, in order to improve the stability of the vibration system, one should have $\tau < \pi/\sqrt{1 - \xi^2}$.

Example 1. To demonstrate the main result obtained in Section 2, let us find out the region D in (k, τ)-plane, under the state difference feedback $-k(x(t) - x(t-\tau))$, ($k > 0$), for $\ddot{x}(t) + 0.12\dot{x}(t) + x(t) = 0$. The controlled system (2) with $\xi = 0.06$ is

$$\ddot{x}(t) + 0.12\dot{x}(t) + x(t) = -k(x(t) - x(t-\tau)) \tag{17}$$

and the corresponding $g(k, \tau, 0)$, $g(k, \tau, 1)$ are given by

$$g(k, \tau, 0) = 0.9964 + k - ke^{0.06\tau}$$

$$g(k, \tau, 1) = -\frac{\pi^2}{\tau^2} + 0.9964 + k + ke^{0.06\tau}$$

The region D, shown in Fig. 2(a), is a small region in (k, τ)-plane. Figs. 2(b)–(d) show three spacial cases that confirm the main results of this paper, where the time histories are calculated by using MATLAB code `dde23` with AbsTol 1e-8.

For the case of acceleration difference feedback, a similar procedure can be made for determining the region D. In this case, however, one should note that the feedback gain k must satisfy $|k| < 1$. If $|k| > 1$, then the closed-loop must be unstable for any given $\tau > 0$, and if $|k| = 1$, then the stability of the vibration system is not guaranteed by the fact "all the characteristic roots have negative real part", because the characteristic roots may have an accumulation point on the imaginary axis.

5. The general case: γ-stability

To achieve a better stability, the real part of the conjugate rightmost roots should be as small as possible. Then, a similar problem for the displacement difference feedback can be described below. For a given number $\gamma \in (\xi, \infty)$, how to find the admissible (k, τ) such that the closed-loop admits a γ-stability, namely all the roots of $p(s) = s^2 + 2\xi s + 1 + k(1 - e^{-s\tau})$ satisfying

$$\text{Re}(s) < -\gamma? \tag{18}$$

To solve this problem, as done above, let

$$f(\lambda) = p(\lambda - \gamma) = \lambda^2 + 2(\xi - \gamma)\lambda + \gamma^2 - 2\xi\gamma + 1 + k(1 - e^{\gamma\tau}e^{-\lambda\tau}) \tag{19}$$

Here, $f(\lambda)$ is a quasi-polynomial with delay-dependent parameters. It is required to find the admissible gains and delays such that the real parts of all the roots of (19) are non-positive. For simplicity, the delay effect on the stability will be

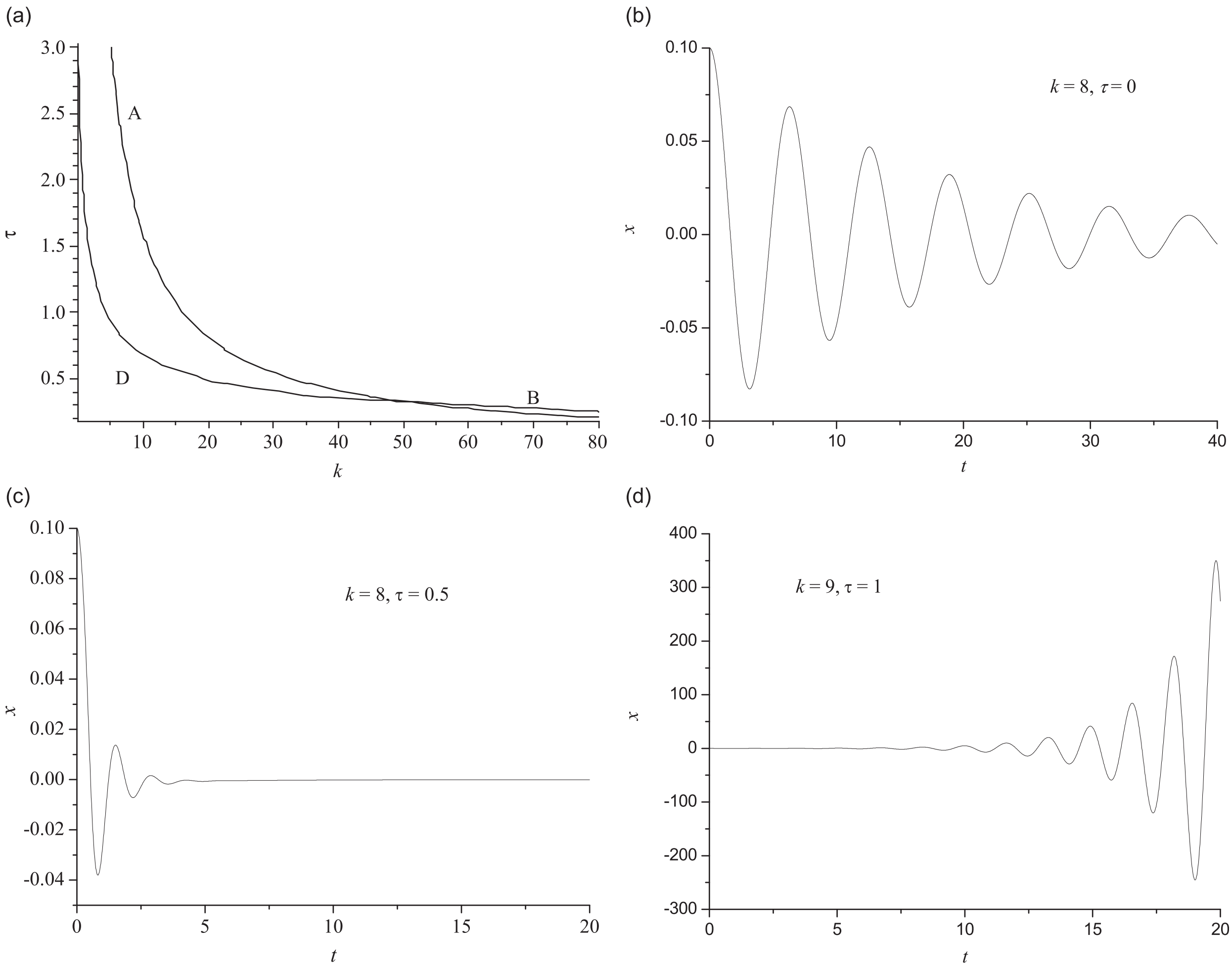

Fig. 2. (a) The region D in (k,τ)-plane that improves the stability of the vibration system for $\xi = 0.06$, where the curve A is the plot of $g(k,\tau,0) = 0$, and the curve B is the plot of $g(k,\tau,1) = 0$. (b) The time history of the uncontrolled system. (c) The time history under control with $(k,\tau) = (8, 0.5) \in D$. (d) The time history under control with $(k,\tau) = (9,1) \notin D$. All start from $x(\theta) = 0.1$, $\dot{x}(\theta) = 0$, $(\forall\theta \in [-\tau, 0])$.

focused on. Firstly, let $\lambda = 0$, then one has

$$\tau = \frac{1}{\gamma}\ln\frac{\gamma^2 - 2\xi\gamma + 1 + k}{k}$$

Then, let $\lambda = \pm i\omega$ with $\omega > 0$, and let the imaginary and real part of (19) equal to zero, then

$$\begin{cases} -\omega^2 + \gamma^2 - 2\xi\gamma + 1 + k - ke^{\gamma\tau}\cos(\omega\tau) = 0 \\ 2\omega\xi - 2\omega\gamma + ke^{\gamma\tau}\sin(\omega\tau) = 0 \end{cases} \tag{20}$$

This is equivalent to the following equations:

$$F(\omega) := \omega^4 + B\omega^2 + C - k^2 e^{2\gamma\tau} = 0 \tag{21}$$

and

$$\begin{cases} \sin(\omega\tau) = 2\omega(-\xi + \gamma)(ke^{\gamma\tau})^{-1} \\ \cos(\omega\tau) = (-\omega^2 + \gamma^2 - 2\xi\gamma + 1 + k)(ke^{\gamma\tau})^{-1} \end{cases} \tag{22}$$

where

$$B = 2\gamma^2 - 2k - 2 - 4\xi\gamma + 4\xi^2$$

$$C = k^2 + (-4\xi\gamma + 2\gamma^2 + 2)k + 1 + 2\gamma^2 - 4\xi\gamma - 4\gamma^3\xi + \gamma^4 + 4\xi^2\gamma^2$$

The number of real roots of the fourth-order polynomial $F(\omega)$ depends on the parameters in three cases. Let

$$\omega_+(\tau) = \sqrt{\tfrac{1}{2}(-B + \sqrt{B^2 - 4(C - k^2 e^{2\gamma\tau})})}$$

$$\omega_-(\tau) = \sqrt{\tfrac{1}{2}(-B - \sqrt{B^2 - 4(C - k^2 e^{2\gamma\tau})})}$$

then, one has

(i) If $B^2 - 4(C - k^2 e^{2\gamma\tau}) < 0$, or $B > 0$ and $C - k^2 e^{2\gamma\tau} > 0$, then none of the roots are real.
(ii) If $C - k^2 e^{2\gamma\tau} < 0$, namely,

$$\tau > \frac{1}{2\gamma}\ln\frac{C}{k^2} = \frac{1}{\gamma}\ln\frac{\gamma^2 - 2\xi\gamma + 1 + k}{k}$$

then $F(\omega)$ has two real roots $\pm\omega_+$.
(iii) If $B^2 - 4(C - k^2 e^{2\gamma\tau}) \geq 0$, $B < 0$ and $C - k^2 e^{2\gamma\tau} \geq 0$, namely, $B < 0$ and

$$\frac{1}{2\gamma}\ln\frac{4C - B^2}{4k^2} \leq \tau \leq \frac{1}{2\gamma}\ln\frac{C}{k^2}$$

then the four roots are all real $\pm\omega_+, \pm\omega_-$. In particular, if

$$\tau = \frac{1}{2\gamma}\ln\frac{C}{k^2}$$

then $\omega_- = 0$, and if

$$\tau = \frac{1}{2\gamma}\ln\frac{4C - B^2}{4k^2}$$

then $\omega_- = \omega_+$.

Once a simple positive root $\omega = \omega(\tau)$ of $F(\omega) = 0$ is in hand, the critical values of delay τ is determined from

$$\tau = \frac{1}{\omega}\left(2n\pi + \operatorname{arccot}\frac{-\omega^2 + \gamma^2 - 2\xi\gamma + 1 + k}{2\omega(-\xi + \gamma)}\right), \quad n = 0, 1, \ldots \tag{23}$$

because $\gamma - \xi > 0$. Let $g(n, \omega)$ be the right hand term of Eq. (23), then $g(0, \omega) < g(1, \omega) < g(2, \omega) < \cdots$, and $\lim_{\omega\to+\infty} g(n, \omega) = 0$. Because $\lim_{\tau\to+\infty} \omega_+(\tau) = \pm\infty$, one has $\lim_{\tau\to+\infty} g(n, \omega) = 0$, thus $\tau > g(n, \omega(\tau))$, for sufficient large τ.

For a given $k > 0$, assuming $(\omega^*(\tau^*), \tau^*)$ is a simple positive solution of (21) and (23) for some n, then (19) has a pair of simple conjugate pure imaginary roots $\lambda_+(\tau^*) = i\omega^*(\tau^*)$ and $\lambda_-(\tau^*) = -i\omega^*(\tau^*)$ at $\tau = \tau^*$. Let

$$\Lambda_{\tau^*} = \operatorname{sign}\left[\operatorname{Re}\frac{d\lambda}{d\tau}\right]_{\lambda=\pm i\omega^*(\tau^*)} = \operatorname{sign}\{R\}$$

where

$$R = \omega^{*4} + (2\xi^2 - \xi\gamma - 1 - k)\omega^{*2} - \gamma^4 + 3\gamma^3\xi - (k + 2\xi^2 + 1)\gamma^2 + (1 + k)\xi\gamma \tag{24}$$

then, $f(\lambda)$ admits one new pair of conjugate characteristic roots with positive real parts if $\Lambda_{\tau^*} = 1 > 0$, and it admits one new pair of conjugate characteristic roots with negative real parts if $\Lambda_{\tau^*} = -1 < 0$.

For Case (i), the ξ-stability is kept unchanged for all given $\tau > 0$. It means that in this case the stability cannot be improved to γ-stability with $\gamma > \xi$.

In Case (ii), the critical delay values can be determined from

$$\tau = g(n, \omega_+(\tau)) \quad (n = 0, 1, 2, \ldots) \tag{25}$$

While in the case of (iii), the critical delay values are determined from

$$\tau = g(n, \omega_+(\tau)), \quad \tau = g(n, \omega_-(\tau)) \;\; (n = 0, 1, 2, \ldots) \tag{26}$$

At

$$\tau = \frac{1}{2\gamma}\ln\frac{C}{k^2} = \frac{1}{\gamma}\ln\frac{\gamma^2 - 2\xi\gamma + 1 + k}{k}$$

$f(\lambda)$ has a repeated root $\omega_- = 0$. As τ passes through

$$\frac{1}{2\gamma}\ln\frac{C}{k^2}$$

a stability switch usually occurs.

Now, for given parameters, the critical delay values in Cases (ii) and (iii) can be found out numerically easily. Consequently, the delay interval that improves the stability of the vibration system to γ-stability can be obtained. The main results could find applications in some fields including machining tool dynamics.

Example 2. With $\xi = 0.02$, $k = 2$ and $\gamma = 0.8$, one has

$$f(\lambda) = \lambda^2 - 1.56\lambda + 3.608 - 2\mathrm{e}^{0.8\tau}\mathrm{e}^{-\lambda\tau}$$

$$F(\omega) = \omega^4 - 4.7824\omega^2 + 13.0177 - 4\mathrm{e}^{1.6\tau}$$

$$R(\omega) = \omega^4 - 3.0152\omega^2 - 2.2514$$

It is easy to know that $F(\omega)$ has real roots only if $\tau > 0.3760$. It means that the stability of the vibration system cannot be improved to γ-stability for all $\tau \in [0, 0.3760]$. In this case, $f(\lambda)$ has a pair of conjugate roots with positive real part. When $0.3760 < \tau < 0.7375$, $F(\omega)$ has two pairs of real roots $\pm\omega_+$, $\pm\omega_-$, and the unique critical delay value is $\tau_{0-} = 0.5329$, determined from $\tau = g(0, \omega_-(\tau))$. Thus, a stability switch cannot occur till $\tau = 0.5329$.

When τ passes though 0.5329, $f(\lambda)$ decreases a pair of conjugate roots with positive real part, because $\Lambda_{\tau_{0-}} = -1$ as seen in Fig. 3(a). Thus, the number of roots with non-negative real parts for $f(\lambda)$ is reduced to 0 as τ passes though 0.5329, namely a stability switch occurs at $\tau = 0.5329$. The number of roots with non-negative real parts for $f(\lambda)$ is kept to be 0 till $\tau = 0.7375$. At $\tau = 0.7375$, $\omega_- = 0$.

If $\tau > 0.7375$, $F(\omega)$ has a pair of real roots $\pm\omega_+$, and the minimal critical delay value is $\tau_{0+} = 0.8736$, determined from $\tau = g(0, \omega_+(\tau))$. As seen in Fig. 3(b), when τ passes though 0.8736, $f(\lambda)$ increases a pair of conjugate roots with positive real part, because $\Lambda_{\tau_{0+}} = 1$. The stability of $f(\lambda)$ cannot be changed till τ passes though 0.8736.

With the Nyquist plot of $f(\mathrm{i}\omega)/(1+\mathrm{i}\omega)^2$, one can easily check whether all the roots of $f(\lambda)$ have negative real parts or not. As proved in [29], if the Nyquist plot does not enclose the origin of the complex plane, then all the roots of $f(\lambda)$ have negative real parts. From Fig. 4 one can see that all the roots of $f(\lambda)$ have negative real parts for all $\tau \in (0.5329, 0.7375)$, and $f(\lambda)$ has one pair of conjugate roots with positive real parts for $\tau \in (0.7375, 0.8736)$. As a result, the stability of the vibration system cannot be improved to γ-stability with $\gamma = 0.8$ for $\tau \notin (0.5329, 0.7375)$, but it can be improved greatly to γ-stability with $\gamma = 0.8$ for $\tau \in (0.5329, 0.7375)$, by using the displacement difference feedback.

Fig. 5 gives the plot of the real part of the rightmost roots with respect to τ for $\xi = 0.02$ and $k = 2$, by following the method proposed in [30]. It shows that the controlled vibration system is stable for $\tau \in (0, 1.4252)$. The real part of the rightmost roots takes its minimal value -1.4793 at about $\tau = 0.63$; it means that the system arrives at its equilibrium most rapidly at about $\tau = 0.63$. Thus, the time delay should be taken around 0.63, if the problem of improving stability is addressed.

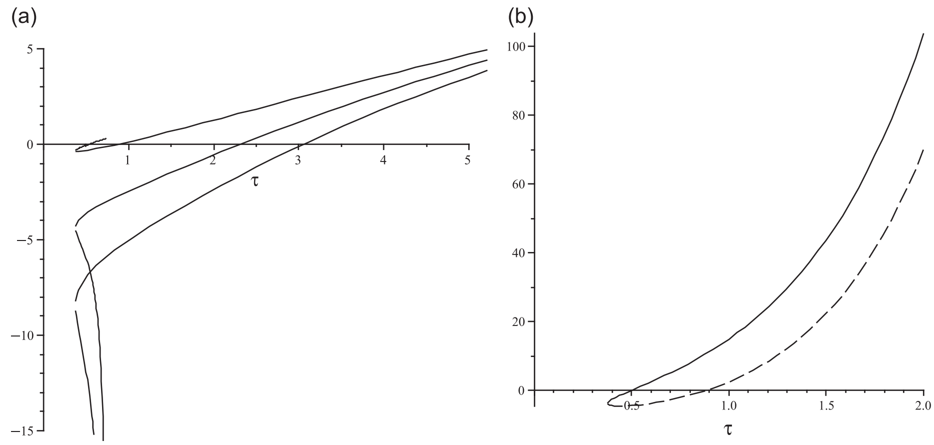

Fig. 3. (a) The first three branches for determining the critical delay values. $\tau = g(0, \omega(\tau))$ gives $\tau_{0-} = 0.5329$, $\tau_{0+} = 0.8736$. (b) The sign of R defined in Eq. (24), for $\xi = 0.02$, $k = 2$ and $\gamma = 0.8$. The solid curve is the plot of $R(\omega_+)$ vs. τ, and the dashed curve is the plot of $R(\omega_-)$ vs. τ. At $\tau = 0.4960$, one has $R(\omega_+) = 0$, and at $\tau = 0.8805$, one has $R(\omega_-) = 0$.

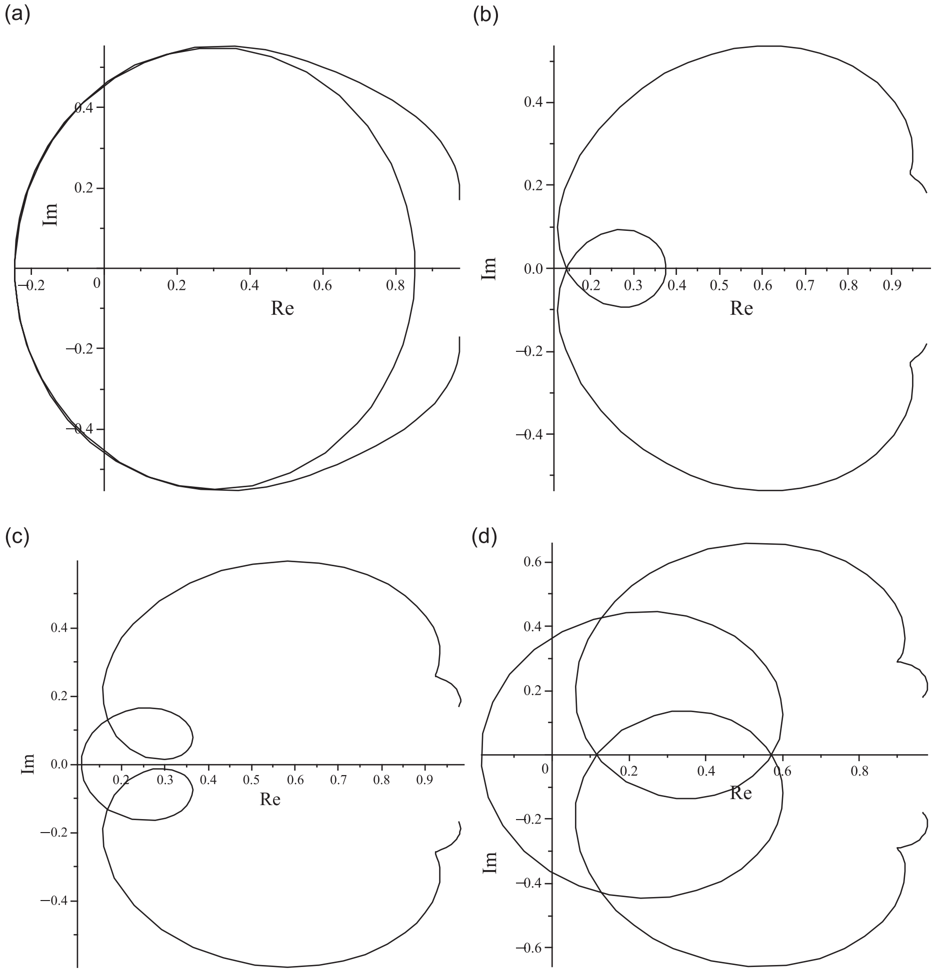

Fig. 4. The Nyquist plot of $f(\mathrm{i}\omega)/(1+\mathrm{i}\omega)^2$, which checks the stability of $f(\lambda)$, for Eq. (19). (a) $\tau = 0.4$; (b) $\tau = 0.6$; (c) $\tau = 0.7$; (d) $\tau = 0.8$. As $\omega \to \pm\infty$, the limit point is $(1, 0)$ for all the cases.

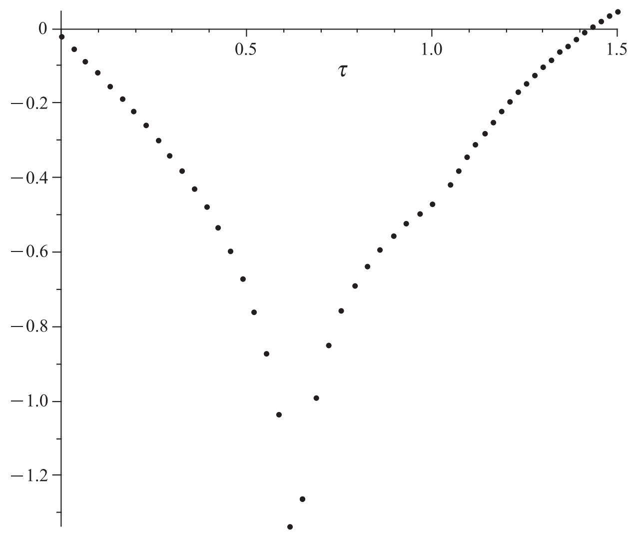

Fig. 5. The real part of the rightmost roots with respect to τ in $\tau \in (0, 1.5)$, for $\xi = 0.02$ and $k = 2$.

6. Concluding remarks

In this paper, the concept of fractional-order difference feedback is proposed for improving the stability of vibration systems. The fractional-order offers us another flexible way in choosing a controller for improving stability of a sdof vibration system. It is found that some fractional-order difference integrators/differentiators improve the stability best. This observation is in agreement with the well-recognized believe: fractional calculus leads to better results than classical calculus.

Because fractional-order controllers involve more complexity in implementation than integer-order controller, and because the optimal fractional-order is close to 0 or 2, the classical displacement difference feedback and the acceleration difference feedback are preferable in applications. If a displacement sensor is used, then the optimal form of state difference feedbacks for enhancing stability of the vibration system with small damping and small delay is the displacement difference feedback with $k>0$. If a acceleration sensor is used, then the optimal form of state difference feedbacks for enhancing stability is the acceleration difference feedback with $k<0$.

On the basis of stability switches, a method is proposed for determining the admissible feedback gains and delay for improving the stability. For the general problem of improving the stability to γ-stability, it is required to determine whether all the roots of a quasi-polynomial have negative real parts or not. The peculiarity of this quasi-polynomial is the dependence of the coefficients on the unknown delay, the procedure for finding the admissible delay is not straightforward. The second numerical example shows that the stability of a slightly damped vibration system can be improved greatly via displacement difference feedback, by adjusting the delay only.

The regenerative chatter instability in machining process, caused by regenerative cutting force or/and regenerative damping force, is unacceptable, and enhancement of dynamic stability must be achieved. Thus, it is expected to find applications of the main results of this paper in machining dynamics.

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